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Space Zeno Effect

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The Zeno effect has been defined and discussed theoretically, and even proved experimentally, only in connection with time-displaced wave functions. This time displacement corresponds to the measurement time of the relevant experiment. If this experiment is repeated successively a very large number of times in a finite time where, in the limit of dense measurement, we take an infinitesimal measurement time, then the initial state is preserved. This is the usual definition of the Zeno effect. In this work the Zeno effect is discussed explicitly in connection with space-displaced wave functions. Here the repetition of the same experiment over the time axis is replaced by simultaneous performances of the same experiment in a number of identical independent nonoverlapping regions of space. We show that when these regions of space shrink infinitesimally (corresponding to the infinitesimal shrinkage of the measurement times in the time Zeno effect), then we obtain a space Zeno effect, that is, the simultaneous performance of such closely spaced experiments has a null effect.

1. INTRODUCTION

The effect of performing the same experiment simultaneously in a very large number of regions of space—all occupying a finite space—is similar to that of performing an experiment repetitively a large number of times in a finite interval of time. The difference is that the repetition in the second case is over independent units of equal steps in time, while in the first case it is over independent units of equal shifts in space. Here we prove that as the Zeno effect (Aharonov and Vardi, 1980; Bar, 1998; Cook, 1988; Giulini *et al.*, 1996; Itano *et al.*, 1990; Misra and Sudarshan, 1977; Chiu *et al.*, 1977; Pascazio and Namiki, 1994; Peres and Ron, 1990; Simonius, 1978; Petrosky *et al.*, 1990, 1991) is obtained in the second case when these equal intervals of time tend to infinitesimal values, so this effect occurs also in the first case when the equal shifts in space tend to be infinitesimal.

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Piron (1979) has discussed a physical example of how this procedure can be seen as an actual evolution. He considers an array of Geiger counters at each of a closely spaced set of points along the x axis. This type of apparatus treats the value of x at which an event occurs as a classical parameter, since the x value of each counter is known in advance. What is unknown is the time t at which the counter will trigger, and this t then becomes a quantum observable. Passing from a counter at x to a counter at $(x + \rho)$ corresponds to a Hamiltonian type evolution $e^{-\frac{ip_1\rho}{\hbar}}$, generated by the evolution operator p_1, now a function on the phase space (H, t, y, z, p_y, p_z). The survival amplitude is $(\phi_{x_0}, e^{-\frac{ip_1\rho}{\hbar}}\phi_{x_0})$, with $\phi_x(t, y, z)$ some "initial" state at position x_0. The successive performances of such experiments along the x axis at small intervals X/n as $n \to \infty$ corresponds precisely to the calculations carried out below, with the interpretation of a joint probability. Since, in this limit, the state ϕ_{x_0} is stabilized (as for the time Zeno effect), the distribution in t, y, and z becomes stationary, and we see that the effect is that of essentially "simultaneous" (at the peak t) measurements over an interval of x.

As pointed by Piron (1979) the two formulations (1) according to the parameter t (as in a bubble chamber type of experiment where t is known, but the locations x, y, and z are subject to measurement) and (2) according to the parameter x (as in the set of Geiger counters described above, where the x is determined, but the times t at which the counters are triggering are the results of measurement) are classically completely equivalent, as can be seen by a change of variables. In the quantum case, the difference is profound, that is, in the first case t is an evolution parameter and x, y, and z are physical observables, while in the second case, x is the parameter of evolution, and t, y, and z are the observables. Our comparison here of the two interpretations of the mathematical equation (11) below corresponds to a qualitative equivalence which carries over, under suitable conditions, to the quantum theory.

It is shown in Hradil *et al.* (1998), by taking the example of a polarized neutron which passes through N identical regions of longitudinal size l in which there is a static magnetic field that induces a rotation of the neutron spin around its axis, that if the parameters connected to this experiment like the size l and the spread δx of the neutron wave packet are taken into account then the limiting Zeno effect may not be obtained. Consider, however, an initial state of a system of N spin-half particles, N_+ of which are in the $1/2$ state, and N_- in the $-1/2$ state, where $N_+ + N_- = N$. If this system is immersed in a rotating field and is left to itself the population difference $N_+ - N_-$ will disappear under the action of the alternating field (Slichter, 1978). It has been done, as shown (Bar, 1999), by two different methods, using either the full Bloch equations or the reduced ones (Slichter, 1978) that if the transversal relaxation time T_t (T_t is the time for a group of spins initially precessing in phase to dephase as a consequence of their slightly different precession frequencies due to their interactions with the environment (Pines and Slichter, 1955)) tends to be very small then the initial state is preserved in spite of the presence of the alternating magnetic field. The condition $T_t \to 0$ is equivalent to performing dense measurement upon the initial system (Bar, 1999).

Moreover, the system discussed in Hradil *et al.* (1998), of a neutron spin passing through N identical regions of longitudinal size l in each of which there is static magnetic field, is based, as stated explicitly in Hradil *et al.* (1998), upon Peres's system of an initial beam of plane polarized light passing through an optically active liquid. This system is discussed in Section 3 of this paper and it is shown that one may view this experiment, as well as that of Hradil *et al.* (1998), mathematically in a different way so that they can be discussed in terms of the space Zeno effect. Hradil *et al.* point out the difficulty in reaching an ideal limit due to the time–energy uncertainty relation. For the type of experiment, to be discussed here there is, indeed, a similar limitation because of the uncertainty principle in x and p; we shall study, however, the case of an ideal limit, and we expect that, nevertheless, one would see an effect experimentally that displays qualitatively the properties of the ideal limit (as in the time-dependent effect observed in Itano *et al.*, 1990; Kofman and Kurizki, 1996; Wilkinson *et al.*, 1997).

As for the simultaneously performed n identical experiments, we do not regard each such experiment as constituting an experiment on its own, only the whole array of all these simultaneously performed identical experiments, all confined to be done in a finite region of space, is considered as an experiment. In the limit of a very large number of such identical experiments, where each one is constrained to take place in an infinitesimal region of space, we have actually a field of them (field of probes, see Bixon, 1982; see also Davies, 1978, 1979).

The same may be said also for the time Zeno effect. That is, the Zeno effect may be discussed not only in terms of the specific experiment that is repeated a very large number of times. We may regard the whole process, composed of the large number of repetitions of the same experiment, as one inseparable process that should not be decomposed, as we have indicated in Eqs. (6)–(7) below. This view is related to that adopted, for example, by Gell-Mann and Hartle (Hartle, 1995; Isham, 1994) in their histories formalism. In this formalism the complete history is regarded as a physical process and the separate parts of it are not considered to be reduced physical entities on their own. It has been shown (Bar, 2000), by discussing the EPR paradox (Aspect, 1988; Bell, 1964; Bohr, 1935; Einstein *et al.*, 1935; Selleri, 1988), the Wheeler's delayed choice experiment (Wheeler, 1983), and the teleportation phenomenon (Bennett *et al.*, 1993) in terms of Feynman paths (Feynman, 1948; Feynman and Hibbs, 1965) that what seems so unique and paradoxical in these three processes is that we conventionally discuss them at their interim stages before they are complete. For example, in the EPR process, we assume that one particle of the two involved has always some definite spin direction even before we measure the spin of the second particle (Einstein *et al.*, 1935);

the EPR experiment is completed only after the latter measurement is performed. Moreover, this effect is remarkable in that it applies to many types of physical phenomena and experiments, even certain classical and macroscopic ones (see Simonius, 1978). Chemical phenomena (see Bixon, 1982, and Harris and Stodolsky, 1981) also have been treated in terms of the Zeno effect. That is, this effect can be discussed solely on the basis of the repetitions without specifying the particular experiment that is repeated. This has been shown in Pati and Lawande (1998) by using the geometric structure of the Fubini–Study metric defined on the projective Hilbert space of the quantum system. With the help of this projective geometry a quantum Zeno effect has been predicted for many types of systems even those described by nonlinear and nonunitary evolution equations, that is, even the linear Schroedinger equation is not a necessary condition.

Nevertheless, all the discussions of the Zeno effect until now, including that of Pati and Lawande (1998), deals explicitly only with the time Zeno effect. We remark in this connection that although the experiment discussed in Hradil *et al.* (1998) refers to a spatial configuration and the limitations imposed by it the set of measurements involved occur in time sequence nevertheless.

In Section 2 we present an example of this space Zeno effect by using the ground state of the harmonic oscillator as our original state, and show that in the limit of space-dense measurement this original state is preserved over the x axis (the more general case is also discussed). In Section 3 we suggest a possible rotation experiment that may test this proposed space version of the Zeno effect. In Section 4 we obtain a result similar to those of Sections 2 and 3, this time for a large ensemble of analyzers in an optically active liquid through which a beam of plane polarized light passes.

2. A COHERENT STATE EXAMPLE OF THE SPACE ZENO EFFECT

We demonstrate now the space Zeno effect for the case of a harmonic oscillator. We consider the ground state of the harmonic oscillator, which is $\phi(x) = (\frac{w}{\hbar\pi})^{\frac{1}{4}}\ e^{-\frac{1}{2\hbar}wx^2}$. Instead of looking at the evolution of this wave function over the time axis, we look at its shifts over a space axis. That is, we look at the wave function at one specific time and see how it varies under displacement on the x axis. The operator that controls this shift is $e^{\frac{i\rho P}{\hbar}}$ (where ρ is the amount by which the wave function is displaced and P is the momentum operator (Schiff, 1968)). The probability to preserve the ground state when the wave function is shifted over the x axis is

$$P(\rho) = \left|\left\langle\phi(x)\middle|e^{\frac{1}{\hbar}i\rho P}\middle|\phi(x)\right\rangle\right|^2$$
$$= \left(\frac{w}{\pi\hbar}\right)^{\frac{1}{2}} \left|\int_{-\infty}^{+\infty} e^{-\frac{1}{2\hbar}wx^2} e^{-\frac{1}{2\hbar}w(x+\rho)^2} dx\right|^2$$

$$= \left(\frac{w}{\pi\hbar}\right)^{\frac{1}{2}} \left| \int_{-\infty}^{+\infty} e^{-\frac{wx^2}{\hbar}} e^{-\frac{1}{\hbar}wx\rho} e^{-\frac{1}{2\hbar}w\rho^2} dx \right|^2$$

$$= \left| \left(\frac{w}{\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{\pi\hbar}{w}\right)^{\frac{1}{2}} \exp\left[\frac{\hbar}{4w}\left(\frac{w^2\rho^2}{\hbar^2} - \frac{2w^2\rho^2}{\hbar^2}\right)\right] \right|^2$$

$$= \left| e^{-\frac{1}{4\hbar}\rho^2 w} \right|^2 = e^{-\frac{w}{2\hbar}\rho^2} \tag{1}$$

The last result is the required probability after doing the experiment once. We now construct an operator, corresponding to the simultaneous performances of the same experiment at neighboring regions, by the projection operator $\mathcal{P} = |\phi\rangle\langle\phi|$ and its translates in steps of spatial amount ρ. The translate of $\mathcal{P}$ is $e^{-\frac{iP\rho}{\hbar}}\mathcal{P}\, e^{\frac{iP\rho}{\hbar}}$. Thus if the same experiment is simultaneously performed twice, separated by ρ, we obtain for the product

$$e^{-\frac{i2P\rho}{\hbar}}\mathcal{P}\, e^{\frac{i2P\rho}{\hbar}} e^{-\frac{iP\rho}{\hbar}}\mathcal{P}\, e^{\frac{iP\rho}{\hbar}} = e^{-\frac{i2P\rho}{\hbar}}\mathcal{P}\, e^{\frac{iP\rho}{\hbar}}\mathcal{P}\, e^{\frac{iP\rho}{\hbar}}. \tag{2}$$

Continuing in this way to the simultaneous measurement for projections shifted n times, each by an amount ρ from the other, we obtain

$$\Pi^{(n)} = e^{-\frac{inP\rho}{\hbar}}\mathcal{P}\, e^{\frac{inP\rho}{\hbar}}\mathcal{P} \cdots e^{\frac{iP\rho}{\hbar}}\mathcal{P}\, e^{\frac{iP\rho}{\hbar}}$$

$$= e^{-\frac{inP\rho}{\hbar}}|\phi\rangle\left(\langle\phi|e^{\frac{iP\rho}{\hbar}}|\phi\rangle\right)^{n-1}\langle\phi|e^{\frac{iP\rho}{\hbar}}, \tag{3}$$

and therefore

$$\Pi^{(n)}|\phi\rangle = e^{-\frac{inP\rho}{\hbar}}\left(\langle\phi|e^{\frac{iP\rho}{\hbar}}|\phi\rangle\right)^n \tag{4}$$

The norm of this result of space-ordered measurements is

$$\left\|\Pi^{(n)}|\phi\rangle\right\| = \left|\langle\phi|e^{\frac{inP\rho}{\hbar}}|\phi\rangle\right|^n \tag{5}$$

We deal here with n ordered, translated, simultaneous performances of the same experiment. Note that the resulting operator $\Pi^{(n)}$ is not a simple projection (filter), but the result of the action of a product ordered not in time, but in space (in three dimensions, for example, the direction is determined by the shift vector ρ).

If these filtering operations are applied in antiordered sequence, one obtains the action of $\Pi^{(n)\dagger} \neq \Pi^{(n)}$. In this case the application of $\Pi^{(n)\dagger}$ to ϕ may vanish if the shift $n\rho$ is larger than the width of the support of $\phi(x)$. However, for example, applying $\Pi^{(n)\dagger}$ to $e^{-\frac{i(n+1)P\rho}{\hbar}}\phi$, the norm of the result is identical to $\|\Pi^{(n)}\phi\|$.

For the usual time Zeno problem, one constructs the sequence

$$\underbrace{\left(\phi, e^{-\frac{iH\delta t}{\hbar}}\phi\right)\cdots\left(\phi, e^{-\frac{iH\delta t}{\hbar}}\phi\right)}_{n} \tag{6}$$

which is, up to a factor $e^{inH\delta t}$, the time-ordered product

$$\Pi_t^{(n)} = e^{\frac{inH\delta t}{\hbar}}\mathcal{P}\,e^{-\frac{inH\delta t}{\hbar}}e^{\frac{i(n-1)H\delta t}{\hbar}}\mathcal{P}\,e^{-\frac{i(n-1)H\delta t}{\hbar}}\cdots e^{\frac{i2H\delta t}{\hbar}}\mathcal{P}\,e^{-\frac{i2H\delta t}{\hbar}}e^{\frac{iH\delta t}{\hbar}}\mathcal{P}\,e^{-\frac{iH\delta t}{\hbar}}, \quad (7)$$

in exact analogy with our construction of the operator, which realizes the space-ordered sequence. We discuss the general properties of operators of this type elsewhere.

Returning to Eqs. (1), (4), and (5) and taking $n = X/\rho$ (X is the finite region of space (in our case the finite section of the x axis) that is composed of all these n displacements, just as T which is the finite overall time composed of the n times by which the experiment is repeated in the time Zeno effect), we obtain that the probability to preserve the system over the x axis is exactly the joint probability $P^n(\rho)$ (which is the norm of the function resulting from this simultaneous measurement)

$$P^n(\rho) = e^{-n\frac{w}{2\hbar}\rho^2} = e^{-X\frac{w}{2\hbar}\rho} \quad (8)$$

Taking the dense measurement limit $n \to \infty$ and $\rho \to 0$ we have

$$\lim_{\rho\to 0} e^{-\frac{Xw\rho}{2\hbar}} \to 1, \quad (9)$$

as in the time Zeno effect. We thus see from (1) and (9) that the original state has been preserved over the x axis, and this is exactly the Zeno effect demonstrated over a space axis, instead of over the time axis. In our treatment we consider the ground state of the harmonic oscillator as a specific example, but, as is obvious from the nature of the Zeno effect, we can generalize our previous equations to a general normalized $\phi(x)$. Suppose the same spatial displacement operator $e^{\frac{i\rho P}{\hbar}}$ acts on our general $\phi(x)$, we use here also the same relation $n = X/\rho$, where n is the large number of the simultaneously performed experiments and X is the finite section of the x axis that is composed of all the n displacements ρ. The probability to preserve $\phi(x)$ as it is shifted over the x axis by the small distance ρ is

$$\begin{aligned}
P(\rho) &= \left|\langle\phi|e^{\frac{1}{\hbar}i\rho p}|\phi\rangle\right|^2 \\
&= \left|1 + \frac{i\rho}{\hbar}\langle\phi|p|\phi\rangle - \frac{\rho^2}{2\hbar^2}\langle\phi|p^2|\phi\rangle\right|^2 \\
&= \left(1 + \frac{i\rho}{\hbar}\langle\phi|p|\phi\rangle - \frac{\rho^2}{2\hbar^2}\langle\phi|p^2|\phi\rangle\right)\left(1 - \frac{i\rho}{\hbar}\langle\phi|p|\phi\rangle - \frac{\rho^2}{2\hbar^2}\langle\phi|p^2|\phi\rangle\right) \\
&= 1 - \frac{\rho^2}{\hbar^2}\langle\phi|p^2|\phi\rangle + \frac{\rho^2}{\hbar^2}(\langle\phi|p|\phi\rangle)^2 = \left(1 - \frac{\rho^2\Delta p^2}{\hbar^2}\right) \quad (10)
\end{aligned}$$

This result is obtained by expanding $e^{\frac{ip\rho}{\hbar}}$ in a Taylor series around ρ and keeping terms up to ρ^2. Also we have denoted the expression $((\langle\phi|p|\phi\rangle)^2 - \langle\phi|p^2|\phi\rangle)$ by

Δp^2. When this experiment is simultaneously performed n times, as described above, then the joint probability for finding the particle in the state ϕ is $(P(\rho))^n$. Using (10), substituting X/n for ρ, and taking the limit $n \to \infty$ so that $\rho \to 0$, we obtain

$$\lim_{\rho \to 0} P^n(\rho) = \lim_{\rho \to 0} \left(1 - \frac{\rho^2 \Delta p^2}{\hbar^2}\right)^n = \lim_{n \to \infty} \left(1 - \frac{X^2 \Delta p^2}{n^2 \hbar^2}\right)^n$$
$$= \lim_{n \to \infty} e^{-\frac{X^2 \Delta p^2}{n \hbar^2}} \to 1 \tag{11}$$

Note that in the former example of the ground state of the harmonic oscillator we obtain this space Zeno effect without having to do any approximate limiting calculations.

It is mathematically straightforward to reformulate the time sequence characteristic of the usual Zeno effect to a description in terms of a simultaneously performed set of measurements in space. What we shall do in the following is to construct physical situations in which one indeed finds the space analog of the time Zeno effect.

3. EXAMPLE OF SPIN ROTATION

Consider in this example a spin 1/2 system characterized by a spinor η, an eigenstate of the spin $\sigma_x/2$ in the x direction. A rotation by $\delta\phi$ of the eigendirection can be achieved by multiplication with the operator

$$U(\delta\phi) = e^{\frac{-i\delta\phi\sigma_z}{2}} \tag{12}$$

The amplitude for finding the system with spin in the x direction after this rotation is

$$A(U(\delta\phi)) = \langle\eta| e^{\frac{-i\delta\phi\sigma_z}{2}} |\eta\rangle \approx 1 - \frac{i\delta\phi}{2}\langle\eta|\sigma_z|\eta\rangle - \left(\frac{\delta\phi}{2}\right)^2 \tag{13}$$

and hence the probability to find this system with spin in the x direction after this rotation is

$$\left|\langle\eta| e^{-i\delta\phi\sigma_z} |\eta\rangle\right|^2 \approx 1 + \left(\frac{i\delta\phi}{2}\right)^2 (\Delta\sigma_z)^2 \tag{14}$$

where

$$(\Delta\sigma_z)^2 = \langle\eta|\sigma_z^2|\eta\rangle - (\langle\eta|\sigma_z|\eta\rangle)^2 = 1 - (\langle\eta|\sigma_z|\eta\rangle)^2$$

We now construct an experiment in which we test the spin system with a compound filter consisting of a large number N of very thin polarizers at slightly different angle $\delta\phi = \phi/N$, where ϕ is some finite angle. The compound filter is represented as

$$\Pi_{\phi}^{(N)} = U^{\dagger}(N\delta\phi)\mathcal{P}_{\eta}U(N\delta\phi)U^{\dagger}((N-1)\delta\phi)\mathcal{P}_{\eta}U \times((N-1)\delta\phi)\cdots U^{\dagger}(2\delta\phi)\mathcal{P}_{\eta}U(2\delta\phi)U^{\dagger}(\delta\phi)\mathcal{P}_{\eta}U(\delta\phi) \tag{15}$$

We see that such a compound filter will produce an effect

$$\Pi_{\phi}^{(N)}|\eta\rangle = e^{iN\delta\phi\frac{\sigma_z}{2}}\left(\langle\eta|e^{-i\delta\phi\frac{\sigma_z}{2}}|\eta\rangle\right)^N \tag{16}$$

so that we obtain as $n \to \infty$

$$\lim_{n\to\infty}\left\|\Pi_{\phi}^{(N)}|\eta\rangle\right\| = \lim_{n\to\infty}\left|\langle\eta|e^{-i\delta\phi\frac{\sigma_z}{2}}|\eta\rangle\right| = 1 \tag{17}$$

One may argue that this compound filter involves a succession of experiments each in a time $\delta t = T/N$, where T is the time for the particle to pass the N filters. The wave function for the system with spin state $|\eta\rangle$ is, however, of the form $\phi_{\eta}(x) = \phi(x) \otimes |\eta\rangle$, and the drift of the system locally is described by the motion of the wave packet defined by $\phi(x)$. This wave packet can be prepared to be wider than the thickness of the compound filter, and therefore the filtering process cannot be put into correspondence with the time sequence of the usual Zeno effect. It corresponds to a simultaneous action of the set of polarizers comprising the compound filter. For example, considering the limit case of a very large number of such polarizers all occupying a finite section along some space axis we must take into account that even in such a limit the width of each polarizer may be effective with width 10–100 Å. If we now take the energy of the wave function to be of the order of 1 eV (from the optical region), then the wavelength associated with such energy is about 12400 Å (Wichman, 1967). Thus, the particle with such an energy and wavelength can be considered as passing simultaneously (at the same time) through all the dense array of these polarizers. If, on the other hand, one thinks of an approximate classical limit in which the wave function for the particle is very small and passes successively through filter after filter, it is easy to see that the Hamiltonian evolution during this succession of measurements has negligible effect on the outcome. The compound filter, therefore, comprises a system in which there is a simultaneous action of "and" filtrations, constituting a space (angle) type Zeno effect.

We, now, describe an example of this space (angle) type Zeno effect. Suppose we divide a finite angle into n equal parts, in each one we have the same kind of interaction that causes the spin of a wave function (as a function of ϕ) to be rotated by the same angle $\delta\phi$. We assume that our wave function have a spin directed with certainty along the x axis at the angle $\phi = 0$, and we want to check the spin state at some angle $\phi = \Phi$. We assume also that the rotation takes place around the z axis. Normally, the state of each wave function is changed in time, so that if we want to graph the evolution of our wave function that passes through the array of n

identical rotating interactions, we need a two dimensional graph in order to depict the real evolution of this wave function. But here we are interested only in the spatial configuration, that is, we cut our two dimensional graph at some specific time point, so that we remain with a one dimensional section of it. The rotation operator has the form (Schiff, 1968)

$$U_R(\phi) = \exp\left(-\frac{i\Phi \cdot J}{\hbar}\right) \tag{18}$$

where J is the total angular momentum of the particle represented by our wave function. We assume here, for simplicity, that our particle is a spin-half particle, that is, the quantum number j is $j = 1/2$. Thus the matrices J are given by $J = 1/2(\hbar\sigma)$, where the σs are the two dimensional Pauli matrices given by (Schiff, 1968)

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

As can be seen from Eq. (18) the rotation Φ is represented by a vector whose direction is the axis about which the rotation takes place, and whose magnitude is the angle of rotation expressed in radians. Since, as we have remarked, this rotation is assumed to occur in the xy plane, the direction of the vector Φ must be along the z axis. Thus Eq. (18), with the help of the third of the last equations and the known characteristics of the matrix exponent (Pipes, 1958), becomes

$$U_R(\phi) = \exp\left(-\frac{i\Phi_z J_z}{\hbar}\right) = \exp\left(-\frac{i\hbar\phi}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} e^{\frac{-i\hbar\phi}{2}} & 0 \\ 0 & e^{\frac{i\hbar\phi}{2}} \end{bmatrix} \tag{19}$$

As we have remarked, all the n interactions in the n parts of the finite region are identical in their action upon the passing wave function. That is, each one rotate the passing wave function by the small angle $\delta\phi$. Thus, remembering that the spin direction of our wave function at the angle $\phi = 0$ was along the x axis, we can write the rotated wave function acted upon by one interaction as

$$|\delta\phi\rangle = \exp\left(-\frac{i\delta\phi J_z}{2}\right)|\delta_x = +1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \exp(i\hbar\delta\phi) \end{bmatrix} \tag{20}$$

where $|\delta_x = +1\rangle$ is $\frac{1}{\sqrt{2}}[\begin{smallmatrix} e^{\frac{i\hbar\delta\phi}{2}} \\ e^{\frac{i\hbar\delta\phi}{2}} \end{smallmatrix}]$ and $|\delta\phi\rangle$ is the eigenstate of the operator

$$\sigma(\delta\phi) = \sigma_x \cos(\delta\phi) + \sigma_y \sin(\delta\phi) \tag{21}$$

with the eigenvalue $+1$. From Eq. (20) we obtain that the probability that the state of the wave function, after the first interaction, remains in its x

direction is

$$P(\sigma_x = 1) = |\langle \sigma_x = 1|\sigma(\delta\phi)\rangle|^2 = |\langle \sigma_x|(|\sigma_x > \cos(\delta\phi) + |\sigma_y\rangle \sin(\delta\phi))|^2$$
$$= \left(\cos\left(\frac{\delta\phi}{2}\right)\right)^2 \approx 1 - \frac{(\delta\phi)^2}{4} \tag{22}$$

The last result is obtained by expanding $(\cos(\frac{\delta\phi}{2}))^2$ in a Taylor series and keeping the first two terms only since it is assumed that $\delta\phi \ll 1$. Now, we perform simultaneously n times the same measurement of σ_x at spatial angular intervals of $\delta\phi$. The probability that we get the same result at each measurement is obtained from (22) as

$$P^n(\sigma_x = 1) = \left(1 - \frac{(\delta\phi)^2}{4}\right)^n \tag{23}$$

The probability we obtain in the last equation is a joint probability. Since we assume that all these n angular measurements are assumed to add to a finite angle Φ in the xy plane we get

$$\delta\phi = \frac{\Phi}{n} \tag{24}$$

If we pass now to the limit of spatial dense measurement $n \to \infty$ we obtain

$$\lim_{n\to\infty} P^n(\sigma_x = 1) = \lim_{n\to\infty} \left(1 - \frac{(\Phi)^2}{n^2}\right)^n = \lim_{n\to\infty} \exp\left(-\frac{1}{n}\right) = 1 \tag{25}$$

Thus we see that we obtain the space Zeno effect. We must point out again that we have here no repetitions of the same measurement as in the time Zeno effect. All these large number of identical measurements done along the finite angle Φ in the xy plane are performed simultaneously and not in a consecutive manner. Each measurement may take a long time to be accomplished. Thus we do not have the problem raised in Hradil *et al.* (1998) of an infinitesimal time allocated to the accomplishment of each measurement. As for the equivalent problem in this space analog of the Zeno effect of the infinitesimal space assigned for each measurement, we have already remarked that at this limit we get simply a field of probes, which has already been discussed in the literature (see, for example, Bixon, 1982, and Harris and Stodolsky, 1981; Pfeifer, 1980; see also Davies, 1978, 1979).

4. A POLARIZATION EXAMPLE OF THE SPACE ZENO EFFECT

The following example was represented by Peres (Peres, 1980) as a demonstration of the time Zeno effect, that is, the effect produced by a large number of repetitions over the time axis. But it is easily shown that this example can also be seen as a demonstration of the space Zeno effect. This is because we have here the same interaction repeated a large number of times over the space axis. Here also, for simplicity, we regard the x axis only. In this example an initial beam of plane polarized light passes through a liquid that is active optically, in such a way that the plane of polarization of the initial beam is rotated by the small angle α. At the end of this liquid we place an analyzer with its optical axis parallel to the initial direction of the polarization. This analyzer may be an ideal Nicol prism. The intensity of the light that passes through the analyzer, according to Malus law (Crawford, 1968) is $I = I_0 \cos^2 \alpha$, where I_0 is the intensity of the incident plane polarized light. The light that passes through the analyzer will have the same polarization as that of the initial beam. If we place a second identical analyzer at the position dividing the optically active liquid into two halves, where the rotation of the original plane of polarization is $\alpha/2$, then the final amount of the light that pass the two analyzers will be $\cos^4(\alpha/2)$. Now for a very small angle α we have $\cos^4(\alpha/2) > \cos^2 \alpha$. In conclusion, if we put a very large number of these ideal analyzers in this liquid, the original plane of polarization will not rotate and all the light will pass through without any diminution in its intensity. We, now, consider a sequence of filtering experiments (as in the actual experiment discussed by Peres (Peres, 1980)) as a sequence in space. For the optical wave, which follows (approximately) a trajectory on the light cone, it is clear that one can replace a time sequence by a spatial one (in light–plane coordinates (Dirac, 1949) this sequence is *simultaneous*).

Let us first consider the general structure of an experiment with a spatial array of analyzers. Let the original wave function be represented by $|\phi(x)\rangle = Ce^{ikx}$, where C is a normalizing constant. We discuss this form of $\phi(x)$ because we do not deal here with a time changing wave functions. Now, the operation on $\phi(x)$ must take into account that we deal with n independent simultaneously performed identical experiments, represented by the n analyzers, where n tends to be a large number. Thus, the nth experiment can be written as

$$E_n = e^{-\frac{ipn\rho}{\hbar}} O_n e^{\frac{ipn\rho}{\hbar}} \tag{26}$$

where O_n is the operator that localizes the wave function Ce^{ikx} at the site of the nth analyser. We shall here approximate the localized reduction of the original wave function by a sharp weight function. We represent these operators by the Gaussian $[\exp(\frac{-(x)^2}{4\sigma_n^2})]/\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}$ where σ_n is the variance of the Gaussian. The factor $e^{\frac{ipn\rho}{\hbar}}$ is the space displacement operator (see Eqs. (1)–(9)) that causes a shifting by ρ to the position of the nth analyser. The n independent simultaneously performed identical experiments can be written as

$$E_T = e^{-\frac{ipn\rho}{\hbar}} \frac{e^{\frac{-(x)^2}{4\sigma_n^2}}}{\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}} e^{\frac{ipn\rho}{\hbar}} e^{-\frac{ip(n-1)\rho}{\hbar}} \frac{e^{\frac{-(x)^2}{4\sigma_n^2}}}{\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}} e^{\frac{ip(n-1)\rho}{\hbar}} \cdots e^{-\frac{ip\rho}{\hbar}} \frac{e^{\frac{-(x)^2}{4\sigma_n^2}}}{\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}} e^{\frac{ip\rho}{\hbar}}$$

$$= e^{-\frac{ip(n-1)\rho}{\hbar}} \left(e^{-\frac{ip\rho}{\hbar}} \left(\frac{e^{\frac{-(x)^2}{4\sigma_n^2}}}{\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}} \right) e^{\frac{ip\rho}{\hbar}} \right)^n \tag{27}$$

The last two equations (26) and (27) are a specific realization and application of the more general equations (3)–(5). Here, we deal with a localizing function rather than a projection. The probability to find the wave equation, after it has passed through all the n analyzers, to be in its original state is

$$P(|\phi(x)\rangle) = |\langle\phi|E_T|\phi\rangle|^2$$

$$= \left| \frac{|C|^2}{\left(\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}\right)^n} \langle e^{-ikx}| e^{-\frac{ip(n-1)\rho}{\hbar}} \left(e^{-\frac{ip\rho}{\hbar}} \left(e^{\frac{-(x)^2}{4\sigma_n^2}} \right) e^{\frac{ip\rho}{\hbar}} \right)^n |e^{ikx}\rangle \right|^2 \tag{28}$$

We, now, use the fact that the application of the space displacement operator $e^{-\frac{ip\rho}{\hbar}}$ to any space-dependent wave function shifts it by ρ (Klauder and Sudarshan, 1968; Schiff, 1968), that is,

$$e^{-\frac{ip\rho}{\hbar}} \left(e^{\frac{-(x)^2}{4\sigma_n^2}} \right) e^{\frac{ip\rho}{\hbar}} = e^{\frac{-(x-\rho)^2}{4\sigma_n^2}} \tag{29}$$

Substituting from Eq. (29) into Eq. (28) we obtain

$$P(|\phi(x)\rangle) = \left| \frac{|C|^2}{\left(\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}\right)^n} \langle e^{-ikx}| e^{-\frac{ip(n-1)\rho}{\hbar}} \left(e^{\frac{-(x-\rho)^2}{4\sigma_n^2}} \right)^n |e^{ikx}\rangle \right|^2 \tag{30}$$

Transforming from the Dirac notation and using the relations (Pipes, 1958)

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)}\, dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2-4ac}{4a}},$$

and (Klauder and Sudarshan, 1968)

$$e^Y e^X = e^{-Z} e^X e^Y,$$

where Z, which is the commutator $Z = [X, Y]$, commutes with both X and Y, we obtain

$$P(|\phi(x)\rangle) = \frac{|C|^4}{\left(\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}\right)^{2n}} \left| \int_{-\infty}^{+\infty} e^{\frac{-ip(n-1)\rho}{\hbar}} e^{ik(n-1)\rho} e^{\frac{-n(x-\rho)^2}{4\sigma_n^2}} dx \right|^2$$

$$= \left(\frac{|C|^4}{\left(\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}\right)^{2n}} \right) \left(\sqrt{\frac{4\pi\sigma_n^2}{n}} \right) \tag{31}$$

We are interested in the limit of very large n, so if we choose σ_n to depend on n such that

$$\sigma_n = \frac{1}{\sqrt{\pi}} \left(\frac{4}{n}\right)^{\frac{1}{2(n-1)}} |C|^{\frac{4}{n-1}},$$

then, $\sigma_n \to 1/\sqrt{\pi}$ from above as $n \to \infty$, and we can write Eq. (31) as

$$\lim_{n\to\infty} P(|\phi(x)\rangle) = \lim_{n\to\infty} \left(\frac{|C|^4}{\left(\sqrt{(\pi)^{\frac{1}{2}}\sigma_n}\right)^{2n}} \right) \left(\sqrt{\frac{4\pi\sigma_n^2}{n}} \right) = 1 \tag{32}$$

We see, therefore, that this example in which we have a very large number of identical physical instruments deployed densely over the x axis in a finite section of it, and all performing the same measurement appears to be an example of the space Zeno effect.

Let us now return to the polarization experiment described by Peres (1980) in terms of the energy–time representation, and convert it to the momentum–position representation. The polarized photons can be described as a system with the two states: $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ as the original state and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ as the decayed state. The eigenvalue equation is given by

$$i\nabla_x \phi = A\phi \tag{33}$$

The coefficients matrix of our eigenvalue equations for this two-dimensional Hilbert space is (compare with Eqs. (36)–(40) for the enlarged three-dimensional Hilbert space)

$$A = \begin{bmatrix} 0 & -ik \\ ik & 0 \end{bmatrix} \tag{34}$$

where the wave function is $\phi = \left[\begin{smallmatrix}\cos kx\\ \sin kx\end{smallmatrix}\right]$. For small x we get the quadratic decay law:

$$|a_0|^2 = \cos^2(kx) \approx 1 - k^2x^2 \tag{35}$$

where a_0 is the survival amplitude. We add now a third state that is connected to the decay product only (we do not assume any additional interaction for the original state). Thus our former two dimensional Hilbert space is enlarged to a three dimensional one. Likewise, the two dimensional matrix from Eq. (34) is enlarged to a three dimensional one

$$A = \begin{bmatrix} 0 & -ik & 0 \\ ik & 0 & -i\Omega \\ 0 & i\Omega & 0 \end{bmatrix} \tag{36}$$

Accordingly, the wave function has three components: $\phi = [\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}]$. The coefficient a is the survival amplitude which we have denoted formerly by a_0. Originally we have $a = 1$, and $b = c = 0$. Using now the eigenvalue equation from Eq. (33) we obtain

$$\begin{aligned} \nabla_x a &= -kb \\ \nabla_x b &= ka - \Omega c \\ \nabla_x c &= \Omega b \end{aligned} \tag{37}$$

If we differentiate the second of equations (Eq. (37)) with respect to x, then using the first and third we get

$$\nabla_{xx} b = -(k^2 + \Omega^2) b \tag{38}$$

where Ω is the coupling with the decay product, which naturally is much stronger than the coupling with the original undecayed state (Slichter, 1978). That is, $\Omega \gg k$, so from Eq. (38) we see that the oscillatory motion of b is much more rapid now than before its interaction with c. The solution of (38) is

$$b = C \sin\left((k^2 + \Omega^2)^{\frac{1}{2}} x\right) \tag{39}$$

C is the constant amplitude. Using (39) we get for the first of (37)

$$\nabla_x a = -kC \sin\left((k^2 + \Omega^2)^{\frac{1}{2}} x\right) \tag{40}$$

Solving the last equation for the survival amplitude a, taking into account that the original survival amplitude was unity, we get

$$a = kC \frac{\cos\left((k^2 + \Omega^2)^{\frac{1}{2}} x\right)}{(k^2 + \Omega^2)^{\frac{1}{2}}} + 1 - \frac{kC}{(k^2 + \Omega^2)^{\frac{1}{2}}} \tag{41}$$

From the last equation we get

$$a_{\max} - a_{\min} = \frac{kC}{(k^2 + \Omega^2)^{\frac{1}{2}}} \tag{42}$$

Since, as we have remarked, $\Omega \gg k$, the last relation tends to 0, thus establishing the space Zeno effect for this polarization example in a way quite analogous to the general space Zeno construction we have described above.

5. CONCLUDING REMARKS

It is shown through different physical examples that there exists a space analog of the conventional Zeno effect. We have shown this for the ground state of the harmonic oscillator, for the general $\phi(x)$, for the rotated wave function, and for the system of a very large number of analyzers immersed in an optically active liquid through which passes a plane polarized light as an example in the more general framework of a space Zeno effect. We have reviewed briefly the viewpoint of Piron (Piron, 1979) in constructing a complementary dynamical picture in which the coordinate plays the role of an evolution parameter. In the examples we discuss here, we show within the framework of the standard formulation of the theory how such an effect can occur. In this version of the Zeno effect the space takes the same role taken by time in the time Zeno effect, and as the last effect is obtained in the limit of dense measurement when the time allocated to each identical repeated (ideal) experiment becomes infinitesimal, so here also this space Zeno effect is obtained in the limit when the space allocated for each experiment becomes infinitesimal.

The conclusion we have obtained is that in such a limit, when the magnitude of each experimental setup becomes very small whereas the total volume (in which all these experiments are simultaneously performed) does not change, we actually get a field of such probes. The known physical fields, such as the electromagnetic field, can be regarded as such fields of probes as has been done by several authors. Bixon (1982) has shown that the stabilization of the localized Born–Oppenheimer states is due to the surrounding medium composed from such a field of probes. In his paper, Bixon himself regards this stabilizing effect of the surrounding field as a manifestation of the Zeno effect, though he regards it as the conventional time Zeno effect and not the space analog of it as discussed in this paper. Davies (1978, 1979) has treated the electromagnetic field as a field of probes and shown that interaction with it causes the localized state to acquire energy lower than the extended state and thus stabilizing it. In such cases it is meaningless to discuss these fields in terms of the separate points as it is meaningless to treat the electromagnetic field pointwise. The same is true also for the time Zeno effect as we have remarked in Section 2. That is, it is necessary to look upon the whole array of these elements of measurements, and from such a perspective the Zeno effect is obtained not only theoretically but also experimentally, as has been done for the time Zeno effect by Itano *et al.* (1990) (see also Kofman and Kurizki, 1996, and Kurizki *et al.*, 1995 that show the validity of this effect with regard to the excitation decay of the atom in open cavities and waveguides using a sequence of pulses on the nanosecond scale); see also Wilkinson *et al.* (1997) for another way of showing experimentally the Zeno effect, this time in quantum tunneling. It appears, therefore, to be a very real effect. In Section 3 we have suggested a possible rotation experiment that may test this space Zeno effect. In Section 4 we have shown that the specific example discussed by Peres in (Peres, 1980) as a representative example of the time Zeno effect may be discussed physically and mathematically as an example of the space Zeno effect.

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